

A version of Krasnoselskii's compression–expansion fixed point theorem in cones for discontinuous operators with applications

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Abstract

We introduce a new fixed point theorem of Krasnoselskii type for discontinuous operators. As an application we use it to study the existence of positive solutions of a second–order differential problem with separated boundary conditions and discontinuous nonlinearities.

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1 Introduction

A classical problem [9, 10, 12] is that of the existence of positive solutions for the differential equation

$$u''(t) + g(t)f(u(t)) = 0, \quad (0 < t < 1), \quad (1.1)$$

along with suitable boundary conditions (BCs).

This problem arises in the study of radial solutions in \mathbb{R}^n , $n \geq 2$ for the partial differential equation (PDE)

$$\Delta v + h(\|x\|)f(v) = 0, \quad x \in \mathbb{R}^n, \quad \|x\| \in [R_1, R_2],$$

with the appropriate boundary conditions, see [4, 9, 10].

Recently, in the paper [8], the authors study the existence of non trivial radial solutions for a system of PDEs of the previous type. First, they turn the former problem into a system of ordinary differential equations similar to (1.1).

The main novelty in this paper is that we will let f to be discontinuous.

The classical compression–expansion fixed point theorem of Krasnoselskii (see [2] or [13]) is a well–known tool of nonlinear analysis and it has proved very useful to deduce existence of solutions for

nonlinear problems. Here we prove a generalization of that theorem which allows discontinuous operators. The idea is similar to that employed in [5, 11], where Schauder's fixed point theorem was extended. Then we return to problem (1.1) along with Sturm–Liouville BCs and we use our extension of Krasnoselskii's theorem to get a result about existence of positive solutions when f is not necessarily continuous.

2 Krasnoselskii's fixed point theorem for discontinuous operators

In the sequel we need the following definitions. A closed and convex subset K of a Banach space $(X, \|\cdot\|)$ is a cone if it satisfies the following conditions:

- (i) if $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$;
- (ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines the partial order in X given by $x \preceq y$ if and only if $y - x \in K$.

Let U be a relatively open subset of K and let $T : \overline{U} \subset K \rightarrow K$ be an operator, not necessarily continuous.

DEFINITION 2.1 *The closed-convex envelope of an operator $T : \overline{U} \rightarrow K$ is the multivalued mapping $\mathbb{T} : \overline{U} \rightarrow 2^K$ given by*

$$\mathbb{T}x = \bigcap_{\varepsilon > 0} \overline{\text{co}} T(\overline{B}_\varepsilon(x) \cap \overline{U}) \quad \text{for every } x \in \overline{U}, \quad (2.2)$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε , and $\overline{\text{co}}$ means closed convex hull.

In other words, we say that $y \in \mathbb{T}x$ if for every $\varepsilon > 0$ and every $\rho > 0$ there exist $m \in \mathbb{N}$ and a finite family of vectors $x_i \in \overline{B}_\varepsilon(x) \cap \overline{U}$ and coefficients $\lambda_i \in [0, 1]$ ($i = 1, 2, \dots, m$) such that $\sum \lambda_i = 1$ and

$$\left\| y - \sum_{i=1}^m \lambda_i T x_i \right\| < \rho.$$

The previous definition was formulated for open subsets of a cone, but it works for arbitrary nonempty subsets of a Banach space (see [11]).

Closed-convex envelopes (cc-envelopes, for short) need not be upper semicontinuous (usc, for short), see [3, Example 1.2], unless some additional assumptions are imposed on T .

PROPOSITION 2.2 *Let \mathbb{T} be the cc-envelope of an operator $T : \overline{U} \rightarrow K$. The following properties are satisfied:*

1. *If T maps bounded sets into relatively compact sets, then \mathbb{T} assumes compact values and it is usc;*
2. *If $T\overline{U}$ is relatively compact, then $\mathbb{T}\overline{U}$ is relatively compact.*

Proof. Let $x \in \overline{U}$ be fixed and let us prove that $\mathbb{T}x$ is compact. We know that $\mathbb{T}x$ is closed, so it suffices to show that it is contained in a compact set. To do so, we note that

$$\mathbb{T}x = \bigcap_{\varepsilon > 0} \overline{\text{co}} T(\overline{B}_\varepsilon(x) \cap \overline{U}) \subset \overline{\text{co}} T(\overline{B}_1(x) \cap \overline{U}) \subset \overline{\overline{\text{co}} T(\overline{B}_1(x) \cap \overline{U})},$$

and $\overline{\overline{\text{co}} T(\overline{B}_1(x) \cap \overline{U})}$ is compact because it is the closed convex hull of a compact subset of a Banach space; see [1, Theorem 5.35]. Hence $\mathbb{T}x$ is compact for every $x \in \overline{U}$, and this property allows us to check that \mathbb{T} is usc by means of sequences, see [1, Theorem 17.20]: let $x_n \rightarrow x$ in \overline{U} and let $y_n \in \mathbb{T}x_n$ for all $n \in \mathbb{N}$ be such that $y_n \rightarrow y$; we have to prove that $y \in \mathbb{T}x$. Let $\varepsilon > 0$ be fixed and take $N \in \mathbb{N}$ such that $\overline{B}_\varepsilon(x_n) \subset \overline{B}_{2\varepsilon}(x)$ for all $n \geq N$. Then we have $y_n \in \overline{\text{co}} T(\overline{B}_\varepsilon(x_n) \cap \overline{U}) \subset \overline{\text{co}} T(\overline{B}_{2\varepsilon}(x) \cap \overline{U})$ for all $n \geq N$, which implies that $y \in \overline{\text{co}} T(\overline{B}_{2\varepsilon}(x) \cap \overline{U})$. Since $\varepsilon > 0$ was arbitrary, we conclude that $y \in \mathbb{T}x$.

Arguments are similar for the second part of the proposition. For every $x \in \overline{U}$ and $\varepsilon > 0$ we have

$$\overline{\text{co}} T(\overline{B}_\varepsilon(x) \cap \overline{U}) \subset \overline{\overline{\text{co}} T \overline{U}},$$

and therefore $\mathbb{T}x \subset \overline{\overline{\text{co}} T \overline{U}}$ for all $x \in \overline{U}$. Hence, $\overline{\overline{\text{co}} T \overline{U}}$ is compact because it is a closed subset of the compact set $\overline{\overline{\text{co}} T \overline{U}}$. \square

Now we recall the fixed point theorem mentioned above (see [13, Theorem 13.D]).

THEOREM 2.3 (KRASNOSELSKII) *Let $r_i \leq R$ ($i = 1, 2$) be positive numbers with $r_1 \neq r_2$ and let $T : \overline{B}(0, R) \cap K \rightarrow K$ be a compact mapping. Suppose that*

- (a) $Tx \not\leq x$ for all $x \in K$ with $\|x\| = r_1$,
- (b) $Tx \not\geq x$ for all $x \in K$ with $\|x\| = r_2$.

Then T has at least a fixed point $x \in K$ such that

$$\min \{r_1, r_2\} < \|x\| < \max \{r_1, r_2\}.$$

In this section we introduce a generalization of the previous theorem which is based on the following idea: given a possibly discontinuous operator T , we build its cc-envelope \mathbb{T} and we prove that it has fixed points by means of the version of Krasnoselskii fixed point theorem for multivalued mappings given by Fitzpatrick–Petryshyn [6]. Then we impose suitable conditions on T which, roughly speaking, guarantee that fixed points of \mathbb{T} are fixed point of T too.

For completeness, we recall [6, Theorem 3.2].

THEOREM 2.4 *Let X be a Fréchet space with a cone $K \subset X$. Let d be a metric on X and let $r_1, r_2 \in (0, \infty)$, $r = \max \{r_1, r_2\}$ and $F : \overline{B}(0, r) \cap K \rightarrow 2^K$ u.s.c. and condensing. Suppose there exists a continuous seminorm p such that $(I - F)(\overline{B}(0, r_1) \cap K)$ is p -bounded. Moreover, suppose that F satisfies:*

1. *there is some $w \in K$ with $p(w) \neq 0$ and such that $x \notin F(x) + tw$ for any $t > 0$ and $x \in \partial_K B(0, r_1)$;*

2. $\lambda x \notin F(x)$ for any $\lambda > 1$ and $x \in \partial_K B(0, r_2)$.

Then F has a fixed point x_0 with $\min\{r_1, r_2\} \leq d(x_0, 0) \leq \max\{r_1, r_2\}$.

We are already in a position to introduce and prove the main results in this section, namely, two extensions of Krasnoselskii fixed point theorem for discontinuous operators.

THEOREM 2.5 *Let $r_i \leq R$ ($i = 1, 2$) with $r_1 \neq r_2$ positive numbers and $T : \overline{B}(0, R) \cap K \rightarrow K$ a mapping such that $T(\overline{B}(0, R) \cap K)$ is relatively compact and*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in \overline{B}(0, R) \cap K, \quad (2.3)$$

where \mathbb{T} is the cc-envelope of T as defined in (2.2).

Suppose that

- (a) $\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ and all $\lambda \geq 1$,
- (b) there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for all $\lambda \geq 0$ and all $x \in K$ with $\|x\| = r_2$.

Then T has at least a fixed point $x \in K$ such that

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

Proof. Notice that the multivalued mapping \mathbb{T} fulfills all the conditions in Theorem 2.4, so there exists a point x such that $x \in \mathbb{T}x$ and

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

Moreover we deduce from (2.3) that $x = Tx$ because $\{x\} \cap \mathbb{T}x = \{x\}$. □

A second result leans on compression–expansion type conditions.

THEOREM 2.6 *Let $r_i \leq R$ ($i = 1, 2$) with $r_1 \neq r_2$ positive numbers and $T : \overline{B}(0, R) \cap K \rightarrow K$ a mapping such that $T(\overline{B}(0, R) \cap K)$ is relatively compact and fulfills condition (2.3).*

Let \mathbb{T} be the cc-envelope of T and suppose that

- (i) $y \not\preceq x$ for all $y \in \mathbb{T}x$ and all $x \in K$ with $\|x\| = r_1$,
- (ii) $y \not\preceq x$ for all $y \in \mathbb{T}x$ and all $x \in K$ with $\|x\| = r_2$.

Then T has at least a fixed point $x \in K$ such that

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

Proof. It suffices to show that all the conditions in Theorem 2.5 are satisfied. First, we show that condition (i) implies condition (a) in Theorem 2.5. Let $x \in K$ be such that $\|x\| = r_1$ and let $\lambda \geq 1$; we have to prove that $\lambda x \notin \mathbb{T}x$. Reasoning by contradiction, we assume that $y = \lambda x \in \mathbb{T}x$. Then we have

$$y - x = (\lambda - 1)x \in K \quad (\text{because } \lambda - 1 \geq 0),$$

and this implies that $y \succeq x$, a contradiction with condition (i).

Now for condition (b) in Theorem 2.5. Once again we use a contradiction argument: we assume that for every $w \in K$ such that $\|w\| \neq 0$ we can find $x \in \partial_K B(0, r_2)$ and $\lambda \geq 0$ such that $x \in \mathbb{T}x + \lambda w$, i.e., there exists $y \in \mathbb{T}x$ such that $x = y + \lambda w$. Hence, $x - y = \lambda w \in K$, a contradiction with (ii). \square

REMARK 2.7 *Condition (2.3) is weaker than continuity, since if T is continuous then $\mathbb{T}x = \{Tx\}$, so (2.3) is trivially satisfied. In addition, it is not difficult to find discontinuous mappings that verify this condition as we show in our next section.*

Neither Theorem 2.5 nor 2.6 remain true if we replace \mathbb{T} by T in the assumptions, as we show in the following example.

EXAMPLE 2.8 *In $X = \mathbb{R}^2$ we consider the cone $K = \mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$.*

Let $0 < r < R$ and define a mapping $T : K \rightarrow K$ in polar coordinates as

$$T(\rho, \theta) = \begin{cases} (0, 0), & \text{if } \rho \neq r, \\ (r, \frac{\pi}{2}), & \text{if } \theta \in [0, \frac{\pi}{4}), \rho = r, \\ (r, 0), & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}], \rho = r. \end{cases}$$

Note that $\mathbb{T}x = \{Tx\} = \{(0, 0)\}$ for all $x \in K$ such that $\|x\| \neq r$ because T is continuous at those points. For points $x = (r, \theta)$, with $\theta \in [0, \pi/2]$, we have three possibilities: if $\theta \in [0, \pi/4)$, then $\mathbb{T}x$ is the segment with endpoints $(0, 0)$ and $(r, \pi/2)$; if $\theta \in (\pi/4, \pi/2]$, then $\mathbb{T}x$ is the segment with endpoints $(0, 0)$ and $(r, 0)$; finally, $\mathbb{T}(r, \pi/4)$ is the triangle with vertices $(0, 0)$, $(r, 0)$ and $(r, \pi/2)$. Therefore,

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in K.$$

Moreover, conditions (i) and (ii) in Theorem 2.6 are satisfied if we replace \mathbb{T} by T (and we take $r_1 = R$ and $r_2 = r$). However, T has no fixed point in $\overline{B}(0, R) \setminus B(0, r)$.

3 Application to Sturm–Liouville problems

We consider the following generalization of equation (1.1) with separated BCs:

$$\begin{aligned} u''(t) + g(t)f(t, u(t)) &= 0 \quad (0 < t < 1), \\ \alpha u(0) - \beta u'(0) &= 0, \\ \gamma u(1) + \delta u'(1) &= 0, \end{aligned} \tag{3.4}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The usual approach to this problem consists in turning it into a fixed point problem with the integral operator

$$Tu(t) := \int_0^1 G(t, s)g(s)f(s, u(s)) ds,$$

where G is the Green's function associated to the differential problem.

Motivated by this situation, we study existence of fixed points of Hammerstein integral operators

$$Tu(t) := \int_0^1 k(t, s)g(s)f(s, u(s)) ds, \tag{3.5}$$

defined in a suitable space. Here we consider $\mathcal{C}([0, 1])$, endowed with the usual supremum norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$.

Fixed points of T will be looked for in the cone

$$K = \left\{ u \in \mathcal{C}([0, 1]) : u \geq 0, \min_{t \in [a, b]} u(t) \geq c \|u\| \right\},$$

where $[a, b] \subset [0, 1]$ and $c \in (0, 1]$. This cone was introduced by Guo and it was intensively employed in recent years, for example, see [7, 9, 12].

We suppose that the terms of the Hammerstein equation (3.5) satisfy the following hypotheses:

(H1) $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is such that:

- (a) Compositions $f(\cdot, u(\cdot))$ are measurable whenever $u \in \mathcal{C}([0, 1])$; and
- (b) For each $r > 0$ there exists $R > 0$ such that $f(t, u) \leq R$ for a.a. $t \in [0, 1]$ and all $u \in [0, r]$.

(H2) g measurable and $g(s) \geq 0$ almost everywhere.

(H3) $k : [0, 1] \times [0, 1] \rightarrow [0, \infty)$ is continuous.

(H4) There exists a measurable function $\Phi : [0, 1] \rightarrow [0, \infty)$ satisfying

$$\Phi g \in L^1(0, 1) \text{ and } \int_a^b \Phi(s)g(s) ds > 0,$$

and a constant $c \in (0, 1]$ such that

$$\begin{aligned} k(t, s) &\leq \Phi(s) \quad \text{for all } t, s \in [0, 1], \\ c \Phi(s) &\leq k(t, s) \quad \text{for all } t \in [a, b], s \in [0, 1]. \end{aligned}$$

REMARK 3.1 *Conditions (H1) – (H4) are similar to those requested in [9] with the exception that we do not require f to be continuous. In addition, our assumptions are more general than those in [10] or [12] where the authors require $g \in L^1(0, 1)$ and $\Phi \in \mathcal{C}([0, 1])$.*

LEMMA 3.2 *If conditions (H1) – (H4) are satisfied, then the operator $T : K \rightarrow K$ introduced in (3.5) is well-defined and maps bounded sets into relatively compact sets.*

Proof. The operator T maps K into K . Indeed, we have

$$\|Tu\| = \max_{t \in [0, 1]} \left\{ \int_0^1 k(t, s)g(s)f(s, u(s)) ds \right\} \leq \int_0^1 \Phi(s)g(s)f(s, u(s)) ds.$$

Moreover,

$$\min_{t \in [a, b]} \{Tu(t)\} \geq c \int_0^1 \Phi(s)g(s)f(s, u(s)) ds.$$

Hence, $Tu \in K$ for every $u \in K$.

Now we prove that if $B \subset K$ is an arbitrary nonempty bounded set, then TB is relatively compact. Let $r > 0$ such that $u \in B$ implies $0 \leq u(t) \leq r$ for all $t \in [0, 1]$, and let $R > 0$ be the constant associated to $r > 0$ by condition (H1) (b). Given $u \in B$, we have

$$\int_0^1 k(t, s)g(s)f(s, u(s)) ds \leq R \int_0^1 \Phi(s)g(s) ds < \infty,$$

so TB is uniformly bounded. To see that TB is equicontinuous, it suffices to show that for every $\tau \in [0, 1]$ and $t_n \rightarrow \tau$, we have

$$\lim_{t_n \rightarrow \tau} \int_0^1 |k(t_n, s)g(s)f(s, u(s)) - k(\tau, s)g(s)f(s, u(s))| ds = 0 \quad \text{uniformly in } u \in B. \quad (3.6)$$

To prove it, we note that for every $u \in B$ we have

$$|k(t_n, s)g(s)f(s, u(s)) - k(\tau, s)g(s)f(s, u(s))| \leq Rg(s) |k(t_n, s) - k(\tau, s)|, \quad (3.7)$$

which tends to zero as n tends to infinity for a.a. $s \in [0, 1]$ because k is continuous in $[0, 1]$. Moreover,

$$Rg(s) |k(t_n, s) - k(\tau, s)| \leq 2R\Phi(s)g(s) \quad \text{for all } n \in \mathbb{N},$$

and $2R\Phi g \in L^1(0, 1)$, by (H4), so the dominated convergence theorem and (3.7) yield (3.6). \square

Moreover suppose that the discontinuities of f allow the operator T to satisfy the condition

$$\{u\} \cap \mathbb{T}u \subset \{Tu\} \quad \text{for all } u \in K \cap \mathbb{T}K, \quad (3.8)$$

where \mathbb{T} is the multivalued mapping associated to T defined in (2.2). Examples of this type of nonlinearities f can be looked up in [5, 11].

LEMMA 3.3 *Suppose that condition (3.8) holds and that*

(I_ρ^1) *There exist $\rho > 0$ and $\varepsilon > 0$ such that $f^{\rho, \varepsilon} < m$, where*

$$f^{\rho, \varepsilon} := \sup_{0 \leq t \leq 1, 0 \leq u \leq \rho + \varepsilon} \left\{ \frac{f(t, u)}{\rho} \right\} \quad \text{and} \quad \frac{1}{m} := \sup_{t \in [0, 1]} \int_0^1 k(t, s)g(s) ds.$$

Then $\lambda u \notin \mathbb{T}u$ for all $u \in \partial_K B(0, \rho)$ and all $\lambda \geq 1$.

Proof. Suppose that there exist $\lambda \geq 1$ and $u \in \partial_K B(0, \rho)$ such that $\lambda u = Tv$ for some $v \in \overline{B}_\varepsilon(u) \cap K$, i.e.,

$$\lambda u(t) = \int_0^1 k(t, s)g(s)f(s, v(s)) ds.$$

Taking the supremum for $t \in [0, 1]$,

$$\lambda \rho \leq \sup_{t \in [0, 1]} \int_0^1 k(t, s)g(s)f(s, v(s)) ds \leq \rho f^{\rho, \varepsilon} \sup_{t \in [0, 1]} \int_0^1 k(t, s)g(s) ds \leq \rho f^{\rho, \varepsilon} \frac{1}{m} < \rho, \quad (3.9)$$

a contradiction.

Given $m \in \mathbb{N}$, it is similarly proved that $\lambda u \neq \sum_{i=1}^m \lambda_i T v_i$ for any $v_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0, 1]$ with $\sum_{i=1}^m \lambda_i = 1$. Hence, $\lambda u \notin \text{co}(T(\overline{B}_\varepsilon(u) \cap K))$.

To see $\lambda u \notin \overline{\text{co}}(T(\overline{B}_\varepsilon(u) \cap K))$ we consider two cases: $\lambda = 1$ and $\lambda > 1$.

If $\lambda = 1$, we have $u \notin \mathbb{T}u$ because $u \neq Tu$ and $\{u\} \cap \mathbb{T}u \subset \{Tu\}$.

If $\lambda > 1$, we obtain from (3.9) that $\lambda \rho \leq \rho$, that in this case suppose a contradiction too. \square

In the sequel we denote

$$V_\rho = \left\{ u \in K : \min_{a \leq t \leq b} u(t) < \rho \right\}.$$

In addition, it is trivial to see that $B(0, \rho) \cap K \subset V_\rho \subset B(0, \rho/c) \cap K$, and V_ρ is a relatively open subset of K (since minimum function is continuous).

LEMMA 3.4 *Suppose that condition (3.8) holds and that*

(I_ρ^0) *There exist $\rho > 0$ and $\varepsilon > 0$ such that $f_{\rho, \varepsilon} > M(a, b)$, where*

$$f_{\rho, \varepsilon} := \inf_{a \leq t \leq b, c(\rho - \varepsilon) \leq u \leq \frac{\rho}{c} + \varepsilon} \left\{ \frac{f(t, u)}{\rho} \right\} \quad \text{and} \quad \frac{1}{M(a, b)} := \inf_{t \in [a, b]} \int_a^b k(t, s) g(s) ds.$$

Then $u \notin \mathbb{T}u + \lambda e$ for all $u \in \partial V_\rho$, all $\lambda \geq 0$ and $e(t) \equiv 1$.

Proof. Suppose there exist $u \in \partial V_\rho$ and $\lambda \geq 0$ such that $u = Tv + \lambda e$ for some $v \in \overline{B}_\varepsilon(u) \cap K$. Then

$$u(t) = \int_0^1 k(t, s) g(s) f(s, v(s)) ds + \lambda.$$

Notice that $\|v\| \leq \|u\| + \varepsilon \leq \frac{\rho}{c} + \varepsilon$ and $\min_{t \in [a, b]} v(t) \geq c\|v\| \geq c(\|u\| - \varepsilon) \geq c(\rho - \varepsilon)$. Therefore, for $t \in [a, b]$

$$u(t) = \int_0^1 k(t, s) g(s) f(s, v(s)) ds + \lambda \geq \int_a^b k(t, s) g(s) f(s, v(s)) ds + \lambda \geq \rho f_{\rho, \varepsilon} \int_a^b k(t, s) g(s) ds + \lambda.$$

Taking the infimum in $[a, b]$ we have

$$\rho \geq \rho f_{\rho, \varepsilon} \inf_{t \in [a, b]} \int_a^b k(t, s) g(s) ds + \lambda > \rho + \lambda,$$

a contradiction because $\lambda \geq 0$.

Given $m \in \mathbb{N}$, it is similar to check that $u \neq \sum_{i=1}^m \lambda_i T v_i + \lambda e$ for any $v_i \in \overline{B}_\varepsilon(u)$ and $\lambda_i \in [0, 1]$ ($i = 1, \dots, m$) with $\sum_{i=1}^m \lambda_i = 1$. Hence,

$$u \notin \text{co} \left(T \left(\overline{B}_\varepsilon(u) \cap K \right) \right) + \lambda e.$$

If we consider two cases: $\lambda = 0$ and $\lambda > 0$, and we work in a similar way than in the previous lemma we obtain that $u \notin \mathbb{T}u + \lambda e$. □

THEOREM 3.5 *Under the hypothesis (H1)-(H4) and (3.8), the Hammerstein integral operator (3.5) has at least a positive fixed point in K if either of the following conditions hold:*

- (a) *There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1/c < \rho_2$ such that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ hold.*
- (b) *There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < \rho_2$ such that $(I_{\rho_1}^1)$ and $(I_{\rho_2}^0)$ hold.*

Proof. It is an immediately consequence of the generalization of Krasnoselskii's Theorem 2.5 together with both lemmas above: Lemma 3.3 and Lemma 3.4. □

REMARK 3.6 *Multiplicity results can be obtained combining previous conditions (see [9]).*

Now we return to the differential BVP (3.4). We will say that u is a solution of that problem if $u \in W^{2,1}([0, 1])$ (i.e, if $u \in \mathcal{C}^1([0, 1])$ and $u' \in \text{AC}([0, 1])$, where $\text{AC}([0, 1])$ denote the absolutely continuous functions space defined in $[0, 1]$) and satisfies (3.4).

The problem (3.4) was widely studied looking for positive solutions [4, 9]. However, the novelty here is to let function f be discontinuous. In [4], the authors consider the problem with $g(t)f(t, u) = h(t, u(t))$ where h is continuous and they use a norm compression–expansion theorem in order to guarantee the existence of solutions. On the other hand, in [9], Lan considers f autonomous and continuous and weaker conditions about g , he even replaces the hypothesis integrable by measurable, but it is necessary that $\int_0^1 \Phi(s)g(s)ds < \infty$. Here, as f can be discontinuous, we will require $g \in L^1(0, 1)$.

We can write the differential problem (3.4) as

$$u(t) = \int_0^1 G(t, s)g(s)f(s, u(s))ds =: Tu(t),$$

where G is the associated Green function, that in this case [9] is given by

$$G(t, s) = \frac{1}{\Gamma} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & \text{if } 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & \text{if } 0 \leq t < s \leq 1, \end{cases} \quad (3.10)$$

and it is non negative.

As $G(t, s) \leq G(s, s)$ for all $s, t \in [0, 1]$, it is possible to choose

$$\Phi(s) = G(s, s) = \frac{1}{\Gamma}(\gamma + \delta - \gamma s)(\beta + \alpha s).$$

Moreover we can choose a, b and c in the following way [9]:

(C1) $a, b \in [0, 1]$ such that $-\beta/\alpha < a < b < 1 + \delta/\gamma$, where we consider $\beta/\alpha = \infty$ if $\alpha = 0$ and $\delta/\gamma = \infty$ if $\gamma = 0$.

(C2) $c = \min \{(\gamma + \delta - \gamma b)/(\gamma + \delta), (\beta + \alpha a)/(\alpha + \beta)\}$.

These choices guarantee that $c\Phi(s) \leq G(t, s)$ for $t \in [a, b]$ and $s \in [0, 1]$.

We shall work, as before, in the cone

$$K = \left\{ u \in \mathcal{C}[0, 1] : u \geq 0, \min_{t \in [a, b]} u(t) \geq c \|u\| \right\}.$$

We allow $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ to have discontinuities over the graphs of the following curves.

DEFINITION 3.7 *We say that $\gamma : [r, s] \subset I = [0, 1] \rightarrow [0, \infty)$, $\gamma \in W^{2,1}([r, s])$, is an admissible discontinuity curve for the differential equation $u'' = -g(t)f(t, u)$ if one of the following conditions holds:*

- (a) $\gamma''(t) = -g(t)f(t, \gamma(t))$ for a.e. $t \in [r, s]$ (then we say γ is viable for the differential equation),
- (b) There exist $\varepsilon > 0$ and $\psi \in L^1(r, s)$, $\psi(t) > 0$ for a.e. $t \in [r, s]$ such that either

$$\gamma''(t) + \psi(t) < -g(t)f(t, y) \text{ for a.e. } t \in I \text{ and all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon], \quad (3.11)$$

or

$$\gamma''(t) - \psi(t) > -g(t)f(t, y) \text{ for a.e. } t \in I \text{ and all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon]. \quad (3.12)$$

In this case we say that γ is inviable.

Working with admissible discontinuity curves involves some technicalities gathered in the next lemma and its subsequent corollaries whose proofs will be omitted because they can be found in [11].

LEMMA 3.8 ([11, LEMMA 4.1]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a, b)$, $g \geq 0$ a.e., and $h > 0$ a.e. in (a, b) .*

For every measurable set $J \subset (a, b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for every $\tau_0 \in J_0$ we have

$$\lim_{t \rightarrow \tau_0^+} \frac{\int_{[\tau_0, t] \setminus J} g(s) ds}{\int_{\tau_0}^t h(s) ds} = 0 = \lim_{t \rightarrow \tau_0^-} \frac{\int_{[t, \tau_0] \setminus J} g(s) ds}{\int_t^{\tau_0} h(s) ds}.$$

COROLLARY 3.9 ([11, COROLLARY 4.2]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a, b)$ be such that $h > 0$ a.e. in (a, b) .*

For every measurable set $J \subset (a, b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \rightarrow \tau_0^+} \frac{\int_{[\tau_0, t] \cap J} h(s) ds}{\int_{\tau_0}^t h(s) ds} = 1 = \lim_{t \rightarrow \tau_0^-} \frac{\int_{[t, \tau_0] \cap J} h(s) ds}{\int_t^{\tau_0} h(s) ds}.$$

COROLLARY 3.10 ([11, COROLLARY 4.3]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $f, f_n : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions on $[a, b]$ ($n \in \mathbb{N}$), such that $f_n \rightarrow f$ uniformly on $[a, b]$ and for a measurable set $A \subset [a, b]$ with $m(A) > 0$ we have*

$$\lim_{n \rightarrow \infty} f'_n(t) = g(t) \text{ for a.a. } t \in A.$$

If there exists $M \in L^1(a, b)$ such that $|f'(t)| \leq M(t)$ a.e. in $[a, b]$ and also $|f'_n(t)| \leq M(t)$ a.e. in $[a, b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

We shall also need the following result.

LEMMA 3.11 *If $M \in L^1(0, 1)$, $M \geq 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^1([0, 1]) : |u'(t) - u'(s)| \leq \int_s^t M(r) dr \text{ whenever } 0 \leq s \leq t \leq 1 \right\},$$

is closed in $\mathcal{C}([0, 1])$ with the maximum norm topology.

Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \rightarrow u$ uniformly in $[0, 1]$, then there exists a subsequence $\{u_{n_k}\}$ which tends to u in the \mathcal{C}^1 norm.

Proof. Let $\{u_n\}$ be a sequence of elements of Q which converges uniformly on $[0, 1]$ to some function $u \in \mathcal{C}([0, 1])$; we have to show that $u \in Q$ and a subsequence $\{u_{n_k}\}$ tends to u in the \mathcal{C}^1 norm.

Since each u_n is continuously differentiable, the Mean Value Theorem guarantees the existence of some $t_n \in (0, 1)$ such that

$$u'_n(t_n) = u_n(1) - u_n(0).$$

This implies the existence of some $K > 0$ such that $|u'_n(t_n)| \leq K$ for all $n \in \mathbb{N}$, because $\{u_n\}$ is uniformly bounded in $[0, 1]$. Hence, for every $n \in \mathbb{N}$ and every $t \in [0, 1]$, we have

$$|u'_n(t)| \leq |u'_n(t) - u'_n(t_n)| + |u'_n(t_n)| \leq \int_0^1 M(s) ds + K,$$

so $\{u_n\}$ is bounded in the \mathcal{C}^1 norm. Moreover, the definition of Q implies that the sequence $\{u'_n\}$ is equicontinuous in $[0, 1]$, so the Ascoli–Arzelà Theorem ensures that some subsequence of $\{u_n\}$, say $\{u_{n_k}\}$, which converges in the \mathcal{C}^1 norm to some $v \in \mathcal{C}^1([0, 1])$. As a result, $u = v$, so u is continuously differentiable in $[0, 1]$ and $\{u_{n_k}\}$ tends to u in the \mathcal{C}^1 norm. In particular, $\{u'_{n_k}\}$ tends to u' uniformly in $[0, 1]$.

Moreover, for $s, t \in [0, 1]$, $s \leq t$, and all $k \in \mathbb{N}$, we have

$$|u'_{n_k}(t) - u'_{n_k}(s)| \leq \int_s^t M(r) dr,$$

and going to the limit as k tends to infinity we deduce that $|u'(t) - u'(s)| \leq \int_s^t M(r) dr$. \square

We are now ready for the proof of the main result in this section.

THEOREM 3.12 *Suppose that f and g satisfy the following hypothesis:*

i. $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is such that:

(a) Compositions $f(\cdot, u(\cdot))$ are measurable whenever $u \in \mathcal{C}([0, 1])$; and

(b) For each $r > 0$ there exists $R > 0$ such that $f(t, u) \leq R$ for a.a. $t \in [0, 1]$ and all $u \in [0, r]$.

ii. There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \rightarrow [0, \infty)$, $n \in \mathbb{N}$, such that for a.a. $t \in I$ the function $u \mapsto f(t, u)$ is continuous on $[0, \infty) \setminus \bigcup_{\{n: t \in I_n\}} \{\gamma_n(t)\}$.

iii. $g \in L^1(0, 1)$ and $g(s) \geq 0$ almost everywhere with $\int_a^b g(s) ds > 0$, where a and b are given in (C1).

Moreover, assume that one of the following conditions hold:

(a) There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1/c < \rho_2$ such that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ hold.

(b) There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < \rho_2$ such that $(I_{\rho_1}^1)$ and $(I_{\rho_2}^0)$ hold.

Then the differential problem with separated BCs (3.4) has at least a positive solution $u \in W^{2,1}([0, 1])$.

Proof. The operator $T : K \rightarrow K$ given by

$$Tu(t) = \int_0^1 G(t, s)g(s)f(s, u(s)) ds$$

is well defined and it maps bounded sets into relatively compact ones, as consequence of Lemma 3.2. In addition, as G is the Green function associated to a second–order homogeneous differential problem, $Tu \in W^{2,1}([0, 1])$ for all $u \in K$. On the other hand, given $u \in B(0, \rho_2/c) \cap K = K_2$, we have $g(t)f(t, u(t)) \in L^1[0, 1]$, and there exists $M(t) \in L^1[0, 1]$ such that

$$h(t, u) := g(t)f(t, u) \leq M(t) \text{ for a.e. } t \in [0, 1] \text{ and all } u \in K_2. \quad (3.13)$$

We consider the set

$$Q = \left\{ u \in \mathcal{C}^1([0, 1]) : |u'(t) - u'(s)| \leq \int_s^t M(r) dr \quad (s \leq t) \right\}, \quad (3.14)$$

which is closed in $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$ by virtue of Lemma 3.11.

Hence, since $TK_2 \subset Q$ and Q is a closed and convex subset of $\mathcal{C}([0, 1])$, we have $\mathbb{T}K_2 \subset Q$.

Now we will prove that

$$\{u\} \cap \mathbb{T}u \subset \{Tu\} \quad \text{for all } u \in K_2 \cap \mathbb{T}K_2. \quad (3.15)$$

To do so, we fix an arbitrary function $u \in K_2 \cap Q$ and we consider three different cases.

Case 1 – $m(\{t \in I_n : u(t) = \gamma_n(t)\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that then T is continuous at u .

The assumption implies that for a.a. $t \in I$ the mapping $h(t, \cdot)$ is continuous at $u(t)$. Hence if $u_k \rightarrow u$ in $K_2 \cap Q$ then

$$h(t, u_k(t)) \rightarrow h(t, u(t)) \quad \text{for a.a. } t \in I,$$

which, along with (3.13), yield $Tu_k \rightarrow Tu$ in $\mathcal{C}(I)$.

Case 2 – $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ for some $n \in \mathbb{N}$ such that γ_n is inviable. In this case we can prove that $u \notin \mathbb{T}u$.

First, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (3.12) holds with γ replaced by γ_n . (The proof is similar if we assume (3.11) instead of (3.12), so we omit it.)

We denote $J = \{t \in I_n : u(t) = \gamma_n(t)\}$, and we deduce from Lemma 3.8 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \rightarrow \tau_0^+} \frac{2 \int_{[\tau_0, t] \setminus J} M(s) ds}{(1/4) \int_{\tau_0}^t \psi(s) ds} = 0 = \lim_{t \rightarrow \tau_0^-} \frac{2 \int_{[t, \tau_0] \setminus J} M(s) ds}{(1/4) \int_t^{\tau_0} \psi(s) ds}. \quad (3.16)$$

By Corollary 3.9 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \rightarrow \tau_0^+} \frac{\int_{[\tau_0, t] \cap J_0} \psi(s) ds}{\int_{\tau_0}^t \psi(s) ds} = 1 = \lim_{t \rightarrow \tau_0^-} \frac{\int_{[t, \tau_0] \cap J_0} \psi(s) ds}{\int_t^{\tau_0} \psi(s) ds}. \quad (3.17)$$

Let us now fix a point $\tau_0 \in J_1$. From (3.16) and (3.17) we deduce that there exist $t_- < \tilde{t}_- < \tau_0$ and $t_+ > \tilde{t}_+ > \tau_0$, t_\pm sufficiently close to τ_0 so that the following inequalities are satisfied for all $t \in [\tilde{t}_+, t_+]$:

$$2 \int_{[\tau_0, t] \setminus J} M(s) ds < \frac{1}{4} \int_{\tau_0}^t \psi(s) ds, \quad (3.18)$$

$$\int_{[\tau_0, t] \cap J} \psi(s) ds \geq \int_{[\tau_0, t] \cap J_0} \psi(s) ds > \frac{1}{2} \int_{\tau_0}^t \psi(s) ds, \quad (3.19)$$

and for all $t \in [t_-, \tilde{t}_-]$:

$$2 \int_{[t, \tau_0] \setminus J} M(s) ds < \frac{1}{4} \int_t^{\tau_0} \psi(s) ds, \quad (3.20)$$

$$\int_{[t, \tau_0] \cap J} \psi(s) ds > \frac{1}{2} \int_t^{\tau_0} \psi(s) ds. \quad (3.21)$$

Finally, we define a positive number

$$\tilde{\rho} = \min \left\{ \frac{1}{4} \int_{\tilde{t}_-}^{\tau_0} \psi(s) ds, \frac{1}{4} \int_{\tau_0}^{\tilde{t}_+} \psi(s) ds \right\}, \quad (3.22)$$

and we are now in a position to prove that $u \notin \mathbb{T}u$. It suffices to prove the following claim:

Claim – Let $\varepsilon > 0$ be given by our assumptions over γ_n and let $\rho = \frac{\tilde{\rho}}{2} \min \{\tilde{t}_- - t_-, t_+ - \tilde{t}_+\}$ be where $\tilde{\rho}$ is as in (3.22). For every finite family $u_i \in \overline{B}_\varepsilon(x) \cap K$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \dots, m$), with $\sum \lambda_i = 1$, we have $\|u - \sum \lambda_i T u_i\| \geq \rho$.

Let u_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T u_i$. For a.a. $t \in J = \{t \in I_n : u(t) = \gamma_n(t)\}$ we have

$$y''(t) = \sum_{i=1}^m \lambda_i (T u_i)''(t) = - \sum_{i=1}^m \lambda_i h(t, u_i(t)). \quad (3.23)$$

On the other hand, for every $i \in \{1, 2, \dots, m\}$ and every $t \in J$ we have

$$|u_i(t) - \gamma_n(t)| = |u_i(t) - u(t)| < \varepsilon,$$

and then the assumptions on γ_n ensure that for a.a. $t \in J$ we have

$$y''(t) = - \sum_{i=1}^m \lambda_i h(t, u_i(t)) < \sum_{i=1}^m \lambda_i (\gamma_n''(t) - \psi(t)) = u''(t) - \psi(t). \quad (3.24)$$

Now for $t \in [t_-, \tilde{t}_-]$ we compute

$$\begin{aligned} y'(\tau_0) - y'(t) &= \int_t^{\tau_0} y''(s) ds = \int_{[t, \tau_0] \cap J} y''(s) ds + \int_{[t, \tau_0] \setminus J} y''(s) ds \\ &< \int_{[t, \tau_0] \cap J} u''(s) ds - \int_{[t, \tau_0] \cap J} \psi(s) ds \\ &\quad + \int_{[t, \tau_0] \setminus J} M(s) ds \quad (\text{by (3.24), (3.23) and (3.13)}) \\ &= u'(\tau_0) - u'(t) - \int_{[t, \tau_0] \setminus J} u''(s) ds - \int_{[t, \tau_0] \cap J} \psi(s) ds + \int_{[t, \tau_0] \setminus J} M(s) ds \\ &\leq u'(\tau_0) - u'(t) - \int_{[t, \tau_0] \cap J} \psi(s) ds + 2 \int_{[t, \tau_0] \setminus J} M(s) ds \\ &< u'(\tau_0) - u'(t) - \frac{1}{4} \int_t^{\tau_0} \psi(s) ds \quad (\text{by (3.20) and (3.21)}), \end{aligned}$$

hence $y'(t) - u'(t) \geq \tilde{\rho}$ provided that $y'(\tau_0) \geq u'(\tau_0)$. Therefore, by integration we obtain

$$y(\tilde{t}_-) - u(\tilde{t}_-) = y(t_-) - u(t_-) + \int_{t_-}^{\tilde{t}_-} (y'(t) - u'(t)) dt \geq y(t_-) - u(t_-) + \tilde{\rho}(\tilde{t}_- - t_-),$$

so if $y(t_-) - u(t_-) \leq -\rho$, then $\|y - u\| \geq \rho$. Otherwise, if $y(t_-) - u(t_-) > -\rho$, then we have $y(\tilde{t}_-) - u(\tilde{t}_-) > \rho$ and thus $\|y - u\| \geq \rho$ too.

Similar computations in the interval $[\tilde{t}_+, t_+]$ instead of $[t_-, \tilde{t}_-]$ show that if $y'(\tau_0) \leq u'(\tau_0)$ then we have $u'(t) - y'(t) \geq \tilde{\rho}$ for all $t \in [\tilde{t}_+, t_+]$ and this also implies $\|y - u\| \geq \rho$. The claim is proven.

Case 3 – $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ only for some of those $n \in \mathbb{N}$ such that γ_n is viable. Let us prove that in this case the relation $u \in \mathbb{T}u$ implies $u = Tu$.

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we denote again by $\{\gamma_n\}_{n \in \mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in I_n : u(t) = \gamma_n(t)\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have

$$u''(t) = \gamma_n''(t) = -h(t, \gamma_n(t)) = -h(t, u(t)),$$

and therefore $u''(t) = -h(t, u(t))$ a.e. in $J = \cup_{n \in \mathbb{N}} J_n$.

Now we assume that $u \in \mathbb{T}u$ and we prove that it implies that $u''(t) = -h(t, u(t))$ a.e. in $I \setminus J$, thus showing that $u = Tu$.

Since $u \in \mathbb{T}u$ then for each $k \in \mathbb{N}$ we can guarantee that we can find functions $u_{k,i} \in \overline{B}_{1/k}(u) \cap K_2$ and coefficients $\lambda_{k,i} \in [0, 1]$ ($i = 1, 2, \dots, m(k)$) such that $\sum \lambda_{k,i} = 1$ and

$$\left\| u - \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i} \right\| < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i}$, and notice that $y_k \rightarrow u$ uniformly in I and $\|u_{k,i} - u\| \leq 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$.

For every $k \in \mathbb{N}$ we have $y_k \in Q$ as defined in (3.14), and therefore Lemma 3.11 guarantees that $u \in Q$ and, up to a subsequence, $y_k \rightarrow u$ in the \mathcal{C}^1 topology.

For a.a. $t \in I \setminus J$ we have that $h(t, \cdot)$ is continuous at $u(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$|h(t, u_{k,i}(t)) - h(t, u(t))| < \varepsilon \quad \text{for all } i \in \{1, 2, \dots, m(k)\},$$

and therefore

$$|y_k''(t) + h(t, u(t))| \leq \sum_{i=1}^{m(k)} \lambda_{k,i} |h(t, u_{k,i}(t)) - h(t, u(t))| < \varepsilon.$$

Hence $y_k''(t) \rightarrow -h(t, u(t))$ for a.a. $t \in I \setminus J$, and then Corollary 3.10 guarantees that $u''(t) = -h(t, u(t))$ for a.a. $t \in I \setminus J$.

Therefore the proof of condition (3.15) is over and we conclude by means of Theorem 3.5. \square

REMARK 3.13 *The differential problem (3.4) contains Dirichlet and Robin problems, so the previous result generalizes the existence results given in [10], because here we allow f be discontinuous.*

Competing interests

The authors declare that they have no competing interests.

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